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Unique Structure of Free Particle Bound States

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Author's contribution

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Abstract

Free particle bound states - which exist only as microscopic systems - are discussed in quantum field theory assuming a QED like Lagrangian with fermion and boson fields. Due to a particular structure, severe boundary conditions can be defined related to geometry, momentum and energy-momentum conservation. Applied to hadrons and atoms, masses or binding energies as well as root mean square radii - which are different by many orders of magnitude between hadrons and atoms - are well described. The fulfillment of a total of about 10 boundary conditions (with three adjustable parameters only) can be considered as a unique and precise test of the mathematical structure of the underlying field theory.

Keywords: Free particle bound state description starting from a QED type Lagrangian with fermion and boson fields; solutions for hadrons and atoms with properties consistent with experimental data; crucial test of the structure from severe boundary conditions.

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1 Introduction

Bound or stationary systems belong to the most interesting objects in physics, since their frequency or binding energy is stable over long periods of time. Their average potential and kinetic energies are related by the virial theorem. Often special boundary conditions are needed, which allow crucial tests of the structure of the underlying theories.

A classical example of a stationary system is the pendulum, which exists in many different forms. This system is characterized by a permanent transformation of potential energy into kinetic energy and vice versus. A rather different type of bound state is found in gravitational systems, for which the gravitational attraction is balanced by a rotation of the system. Examples are the orbiting of planets around a star or satellites surrounding the earth. In addition to the virial theorem, momentum conservation $p_m + p_M = 0$ requires that the radial moment of the (lighter) rotating body $p_m = r_m * mv_m$ is compensated by that of its massive partner $p_M = \tilde{r}_M * Mv_M$, where v_x are the rotation velocities.

There is another important type of rotational bound state, free particles in the vacuum, which are not coupled to other pieces of matter. Such states can be found only in microscopic systems in the form of atoms, nucleons or even simpler hadrons built out of elementary particles. Importantly, for these systems binding of fermions is not sufficient, accompanying bosons (not only a boson-exchange interaction) are essential to equilibrate the momentum of fermions. This can be seen in the following way: to satisfy the virial theorem, the fermion potential energy is accompanied by kinetic energy in form of rotation. But this motion would be spurious, if the momentum of the participating fermions could not be counterbalanced by neutral particles (bosons), which don't contribute to the potential energy. This indicates that these systems must have a double bound state structure of fermions and bosons, as shown schematically in Fig. 1.

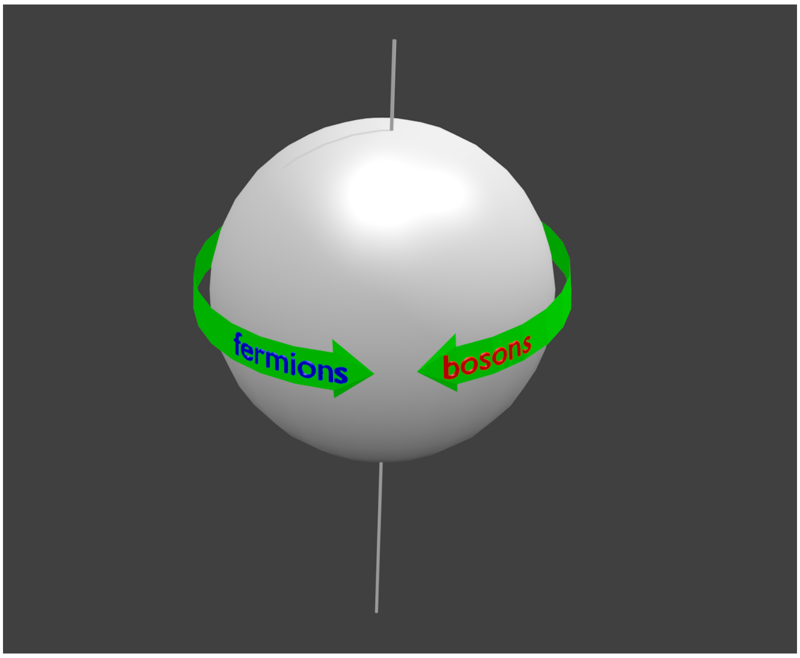

Fig. 1. Schematic view of a free particle bound state, in which the rotation of fermions is counterbalanced by bosons

2 Theoretical Description

For microscopic systems a quantum description is essential, which includes relativity and general transformation invariances, as realized in field theory [1]. In the present discussion we start from quantum electrodynamics (QED) by modifying the Lagrangian to include a double structure of fermions and bosons as shown in Fig. 1. This leads to a gauge invariant Lagrangian given by

$$\mathcal{L} = \frac{1}{\tilde{m}^2}(\bar{\Psi} D_\nu)\, i\gamma^\mu D_\mu\, (D^\nu \Psi) \; - \; \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \; , \tag{2.1}$$

where $\tilde{m}$ is the mass parameter and Ψ are charged fermion fields, $\Psi = \Psi^+$ and $\bar{\Psi} = \Psi^-$. Vector boson fields A_μ (photons) with coupling g to fermions are contained in the covariant derivatives $D_\mu = \partial_\mu - igA_\mu$. The second term of the Lagrangian represents a Maxwell term with Abelian field strength tensors $F^{\mu\nu}$ given by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, which gives rise to both electric and magnetic type coupling.

Due to three covariant derivatives $D_\mu = \partial_\mu - igA_\mu$ the Lagrangian (2.1) leads to a higher order field theory with higher derivatives of boson and fermion fields. Note that there are two strong arguments against the use of higher order Lagrangians. First, the necessary $1/\tilde{m}^n$ factor gives rise to uncontrolled divergences in standard (divergent) field theory, see e.g. ref. [1]. Second, Lagrangians with higher order fermion fields will lead to nonphysical solutions [2]. However, for the present Lagrangian both arguments do not apply: Eq. (2.1) leads to a finite theory due to a constrained normalization of wave functions, for which a $1/\tilde{m}^2$ factor is acceptable and leads to realistic solutions. Further, in the present formalism nonphysical solutions can be excluded by strict boundary conditions, as discussed below.

By inserting $D^\mu = \partial^\mu - igA^\mu$ and $D_\nu D^\nu = \partial_\nu \partial^\nu - ig(A_\nu \partial^\nu + \partial_\nu A^\nu) - g^2 A_\nu A^\nu$ in eq. (2.1), the first part of $\mathcal{L}$ gives rise to a number of terms, which contain boson and fermion fields and/or derivatives. All terms containing the derivative of the fermion field $\partial^\nu \Psi$ are related to a complex dynamics of the system as a whole. For stationary solutions only two terms of the Lagrangian contribute

$$\mathcal{L}_{2g} = \frac{-ig^2}{\tilde{m}^2}\; (\bar{\Psi} A_\nu)\; \gamma^\mu \partial_\mu\; (A^\nu \Psi) \tag{2.2}$$

and

$$\mathcal{L}_{3g} = \frac{-g^3}{\tilde{m}^2}\; (\bar{\Psi} A_\nu)\; \gamma^\mu A_\mu\; (A^\nu \Psi)\; . \tag{2.3}$$

From the Lagrangians (2.2) and (2.3) fermion matrix elements with boson propagators have been derived by using standard procedures, as the evaluation of fermion ground state expectation values or generalized Feynman diagrams, see e.g. ref. [1]. According to these rules the matrix elements can be written in the form $\mathcal{M}^f =< g.s.|\; K(p'-p)\; |g.s. >\sim \bar{\psi}(p')\; K(q)\; \psi(p)$, where $\psi(p)$ is a fermion wave function $\psi(p) = \frac{1}{\tilde{m}^{3/2}} \Psi(p_1)\Psi(p_2)$ and $K(q)$ a kernel related to the boson structure of the Lagrangian. In the present case a kernel is needed of the form $K(q) = \frac{1}{\tilde{m}^5}\; [O^3(q_i)\; O^3(q_j)]$, where $O^3(q_i)$ represents a product of boson fields or derivatives, as given in eqs. (2.2) and (2.3). Using $\alpha = g^2/4\pi$ this leads to matrix elements of the form

$$\mathcal{M}_{2g} = \frac{\alpha^2}{\tilde{m}^5}\; \bar{\psi}(p')\; A_\nu(q_4') A^\nu(q_3')\; \gamma_\mu \gamma^\rho\; \partial A_\sigma(q_2')\; \partial A^\sigma(q_1')\; \psi(p) \tag{2.4}$$

and

$$\mathcal{M}_{3g} = \frac{-\alpha^3}{\tilde{m}^5}\; \bar{\psi}(p')\; A_\nu(q_4') A^\nu(q_3')\; \gamma_\mu \gamma^\rho A_\mu(q_2) A^\rho(q_1)\; A_\sigma(q_2') A^\sigma(q_1')\; \psi(p)\; . \tag{2.5}$$

One may compare these matrix elements to similar ones derived from a first order Lagrangian $\mathcal{L}_{f.o.} = \bar{\Psi}\; i\gamma_\mu D^\mu \Psi \; - \; \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ as used in QED. By writing in a similar way $\mathcal{M} = \bar{\psi}(p')\; K(q)\; \psi(p)$, with a kernel $K(q) = \frac{1}{\tilde{m}}\ [O^1(q_2)\ O^1(q_1)]$ one obtains for the case $\partial\Psi = 0$ only one (boson-exchange) matrix element $\mathcal{M}_{f.o.} = \frac{-\alpha}{\tilde{m}}\ \bar{\psi}(p')\ \gamma_\mu\gamma^\rho A_\mu(q_2)A^\rho(q_1)\ \psi(p)$.

The comparison of both theories shows two essential differences: 1. The "boson-exchange" matrix element $\mathcal{M}_{3g}$ has a more complex structure than $\mathcal{M}_{f.o.}$ with additional boson fields, needed to balance the fermion motion. 2. A second matrix element $\mathcal{M}_{2g}$ is present, which does not exist in first order theories. This term leads to a dynamical stabilization of the system, as discussed below.

From these matrix elements bound state potentials can be deduced. First, the γ-matrices can be eliminated in the usual way by adding a matrix element with interchanged indices, according to the relation $\frac{1}{2}(\gamma_\mu\gamma_\rho + \gamma_\rho\gamma_\mu) = g_{\mu\rho}$. Then, by using a gauge condition $\partial^2 A^\nu = 0$ the product $\partial A_\sigma(q_2')\ \partial A^\sigma(q_1')$ in eq. (2.4) can be replaced by $\frac{1}{2}\partial^2(A_\sigma(q_2')A^\sigma(q_1'))$. Further, (analogue to the fermion wave functions) normalized boson (quasi) wave functions of scalar ($\mu = \nu$) and vector ($\mu \neq \nu$) structure $W_\mu^\nu(q') = \frac{1}{\tilde{m}} A_\mu(q_j')A^\nu(q_i')$ have been introduced, as well as a boson-exchange interaction $V_\mu^\nu(q) = \frac{1}{\tilde{m}} A_\mu(q_2)A^\nu(q_1)$ ($\mu \neq \nu$), which is similar to that of first order QED.

By equal time requirement the fermion and boson vectors can be reduced by one dimension, yielding boson wave functions[1] of scalar and vector structure $w_s(q')$ and $w_v(q')$ and an interaction $v_v(q)$. This leads to

$$\mathcal{M}_{2g} = \frac{\alpha^2}{2\tilde{m}^3}\ \bar{\psi}(p')\ w_s(q')\ \partial^2 w_s(q')\ \psi(p) \tag{2.6}$$

and

$$\mathcal{M}_{3g} = \frac{-\alpha^3}{\tilde{m}^2}\ \bar{\psi}(p')\ w_{s,v}(q')v_v(q)w_{s,v}(q')\ \psi(p)\ . \tag{2.7}$$

The bosonic part of eq. (2.7) can also be written in the form of a matrix element, in which the wave functions $w(q')$ are connected by $v_v(q)$

$$\mathcal{M}^g = \frac{-\alpha^3}{\tilde{m}^2}\ w_{s,v}(q')\ v_v(q)\ w_{s,v}(q'). \tag{2.8}$$

In the following an evaluation of these matrix elements is discussed, for which the Hamiltonian formalism can be used by relating kinetic and potential energies by $(T+V)\psi = E\psi$, where E is the binding energy. For these systems all quantities described in momentum space can be transformed to r-space by Fourier transformation. Energies, masses and momenta are given in energy units (using c=1), whereas the constant $\hbar c$ is used for radius-momentum conversion.

Going to r-space the fermion matrix element (2.6) can be written by

$$\mathcal{M}_{2g}(r) = \bar{\psi}(r)\ V_{2g}(r)\ \psi(r)\ , \tag{2.9}$$

in which $V_{2g}(r)$ is a potential, which can be derived from a boson Hamiltonian of the form

$$-\frac{\alpha^2(\hbar c)^2}{2\tilde{m}}\ \Big(\frac{d^2 w_s(r)}{dr^2} + \frac{2}{r}\frac{dw_s(r)}{dr}\Big) + V_{2g}(r)\ w_s(r) = E_o\ w_s(r)\ . \tag{2.10}$$

This leads to

$$V_{2g}(r) = \frac{\alpha^2(\hbar c)^2}{2\tilde{m}}\ \Big(\frac{d^2 w_s(r)}{dr^2} + \frac{2}{r}\frac{dw_s(r)}{dr}\Big)\frac{1}{w_s(r)} + E_o\ . \tag{2.11}$$

A coupling to the vacuum is made by assuming $E_o = 0$. This implies that the vacuum is the lowest state with energy $E_{vac} = 0$. An important consequence of this is that the elementary fermions (quantons) have to be massless. This potential leads to a dynamical stabilization of the system: with positive eigenvalues, created fermion-antifermion pairs are locked during overlapping boson

[1] with dimension $[GeV]$.

fields and form a stable system, which cannot decay. Remarkably, $V_{2g}(r)$ shows a quite linear rise towards larger radii, very similar to the empirical "confinement" potential required in hadron potential models [3].

Fourier transformation of the matrix element (2.7) leads to

$$\mathcal{M}_{3g}(r) = \bar{\psi}(r)\ V_{3g}(r)\ \psi(r)\ , \tag{2.12}$$

in which the potential $V_{3g}(r)$ has the form of a folding potential

$$V_{3g}^{s,v}(r) = \frac{\alpha^3 \hbar c}{\tilde{m}} \int dr'\ w_{s,v}(r')\ v_v(r - r')\ w_{s,v}(r') \tag{2.13}$$

with an interaction $v_v(r) \sim -\hbar c\ w_v(r)$.

As an important point, the potential (2.13) can be considered also as boson matrix element, in which the bosons are "bound" in the potential $v_v(r)$. It is important to note that due to the condition $E_o = 0$ both potentials (2.11) and (2.13) yield absolute binding energies, for which no constant can be added (this is different from other potentials, as e.g. the harmonic oscillator potential).

A natural condition of a double bound state of fermions and bosons requires that the radial form of the fermion and boson wave functions is similar. Therefore, the existence of two boson wave functions of scalar and vector structure implies that also corresponding fermion wave functions of similar form exist

$$\psi_{s,v}(r) \sim w_{s,v}(r)\ . \tag{2.14}$$

This gives rise to two bound states (scalar and vector) without angular momentum (L=0) and $J^\pi = 1^-$. In addition, two states with angular momentum L=1 (coupled to $J^\pi = 0^+$) are predicted, which are not considered here.

For fermion vector coupling normally an angular distribution of dipole form $P_1^2(cos\theta)$ is needed, but for a free particle without preferred orientation in space there is no angular dependence. However, in the Fourier transformation a Bessel function ${j_1}^2(qr)$ is still required. Orthogonality of these two wave functions leads to the constraint

$$\int r^2 dr\ \psi_s(r)\psi_v(r) = \int r^2 dr\ w_s(r) w_v(r) = < r_{w_s,w_v} >= 0\ . \tag{2.15}$$

This condition can be satisfied only, if the wave functions are finite (with finite radial moments). Condition (2.15) is satisfied for

$$w_v(r) = w_{v_o}\ [w_s(r) + \beta R\ \frac{dw_s(r)}{dr}]\ , \tag{2.16}$$

where w_{v_o} is obtained from the normalization of the density $w_v^2(r)$ with $2\pi \int rdr\ w_v^2(r) = 1$ and $\beta R = -\int r^2 dr\ w_s(r) / \int r^2 dr\ [dw_s(r)/dr]$. Because of the derivative structure, $w_v(r)$ has a smaller root mean square radius than $w_s(r)$. Therefore, a natural condition requires that the interaction for this state takes place inside the bound state volume of the density $w_s^2(r)$. This leads to the geometrical boundary condition

$$|V_{3g}^v(r)| \simeq c\ w_s^2(r)\ . \tag{2.17}$$

The conditions (2.15) and (2.17) can be satisfied by a boson wave function of the scalar state of the form

$$w_s(r) = w_{s_o}\ exp\{-(r/b)^\kappa\}\ , \tag{2.18}$$

where w_{s_o} is fixed by the density normalization $2\pi \int rdr\ w_s^2(r) = 1$. This may be the simplest radial form with slope and shape parameters b and κ, which have to be determined from basic constraints.

Concerning the potentials $V_{2g}(r)$ and $V_{3g}(r)$, $V^s_{3g}(r)$ gives rise to binding of the scalar state, whereas $V^v_{3g}(r)$ contributes only to the vector state. Differently, $V_{2g}(r)$ contributes to scalar and vector states in the ratio 1:3, yielding $V^s_{2g}(r) = \frac{1}{4}V_{2g}(r)$ and $V^v_{2g}(r) = \frac{3}{4}V_{2g}(r)$.

Binding energies have been calculated by use of the virial theorem (in radial form) $E^{ng}_f = 4\pi\ [\int r^2dr\ \psi^2(r)V_{ng}(r) - \frac{1}{2}\int r^3dr\ \psi^2(r)\frac{d}{dr}V_{ng}(r)]$, where the fermion wave functions $\psi(r)$ are normalized by $4\pi\int r^2dr\,\psi^2(r) = 1$. In addition, $V_{3g}(r)$ can be interpreted as "bound state" potential of bosons. The corresponding energies E_g have been calculated by $E_g = 2\pi\ [\int rdr\ w^2(r)v_v(r) - \frac{1}{2}\int r^2dr\ w^2(r)\frac{d}{dr}v_v(r)]$. The masses (due to binding) are defined by the sum of absolute binding energies $M_{s,v} = |E^{s,v}_{2g}| + |E^{s,v}_{3g}|$, and the total mass of the system is given by $M_{tot} = M_{s,v} + m_1 + m_2$, where m_1 and m_2 are the participating fermion masses.

As discussed above, momentum conservation requires that the average fermion momentum of both states is counterbalanced by the momentum of bosons

$$< q^2_g >^{1/2} + < q^2_f >^{1/2} = 0\ . \tag{2.19}$$

The average momentum squares are used in the most natural form by using normalizations for bosons $< q^0_g > = \int qdq\ V_{3g}(q)$ and $< q^0_f > = \int q^2dq\ \psi(q)V_{3g}(q)$ for fermions. This leads to $< q^2_g > = \int q^3dq\ V_{3g}(q)/ < q^0_g >$ and $< q^2_f > = \int q^4dq\ \psi(q)V_{3g}(q)/ < q^0_f >$. The Fourier transformed wave functions and potentials are given by $\psi(q) = 4\pi\int r^2dr\ \psi(r)j_o(qr)$ and $V_{3g}(q) = 4\pi\int r^2dr\ V_{3g}(r)j_o(qr)$, respectively. This condition should be valid for both scalar and vector states.

Another condition can be derived from the structure of the potential $V_{2g}(r)$, which may be written in a different form (this can be seen from dimensional arguments)

$$V_{2g}(r) = \frac{\alpha^2(M_s/2) < r^2_{w_s} >}{2}\ \Big(\frac{d^2w_s(r)}{dr^2} + \frac{2}{r}\frac{dw_s(r)}{dr}\Big)\frac{1}{w_s(r)}\ . \tag{2.20}$$

Dividing both potentials (2.11) and (2.20) leads to a mass-radius constraint

$$Rat_{2g} = \frac{(\hbar c)^2}{\tilde{m}(M_s/2) < r^2_{w_s} >} = 1\ . \tag{2.21}$$

This condition is very powerful and relates all parameters of a system. However, it can also be used to relate different systems.

Finally, for a coupling to the vacuum ($E_o = 0$) energy-momentum conservation[2] should be respected, which requires that the average momenta are compensated by the corresponding binding energies. Since there are separate potentials for fermions and bosons, this gives rise to four additional conditions. For bosons this yields

$$< q^2_g >^{1/2} + E_g = 0\ , \tag{2.22}$$

whereas for fermions one obtains

$$< q^2_f >^{1/2} - x\ M_f = 0\ , \tag{2.23}$$

where $x\ M_f = \sqrt{2\tilde{m}M_f}$ takes into account mass parameter and fermion binding energy in $V_{2g}(r)$ and $V_{3g}(r)$. For the vector state $< q^2_f >_v$ is given by a similar expression as $< q^2_f >_s$ but with Fourier transformed wave functions and potentials given by $\psi_v(q) = 4\pi\int r^2dr\ \psi_v(r){j_1}^2(qr)$ and $V^v_{3g}(q) = 4\pi\int r^2dr\ V^v_{3g}(r){j_1}^2(qr)$, respectively.

[2]energy-momentum conservation is also required from relativistic kinematics.

3 Bound States of Elementary Fermions (Hadrons)

First, an application of the present formalism is discussed for a fermion-antifermion system composed of elementary fermions (quantons), which are massless. In this case one can require further that the mass parameter $\tilde{m}$ is half of the generated bound state mass

$$\tilde{m} = \frac{1}{2} M_s = \frac{1}{2}(|E^s_{2g}| + |E^s_{3g}|) \ . \tag{3.1}$$

By the different boundary conditions (2.14), (2.17), (2.19), (2.21) - (2.23) and (3.1) the slope and shape parameters b and κ as well as the coupling constant α are tightly related to the binding energies (or masses). Solutions have been found, in which all boundary conditions are fulfilled; however, due to the mass-radius ambiguity in eq. (2.21) several parameter choices are possible. One solution is obtained with shape parameter $\kappa = 1.35$, coupling constant $\alpha = 2.14$ and slope parameter $b = 0.572$, which gives rise to a mass of the lowest state of 0.78 GeV, corresponding to the mass of the $\omega(780)$ meson.

Resulting distributions of scalar density and potentials are shown in Fig. 2. In the upper part the potential $V_{2g}(r)$ is given, which has a linearly rising form at larger radii. Importantly, the very special radial dependence of $V_{2g}(r)$ shows the same characteristics as the empirical "confinement" potential needed in hadron potential models [3].

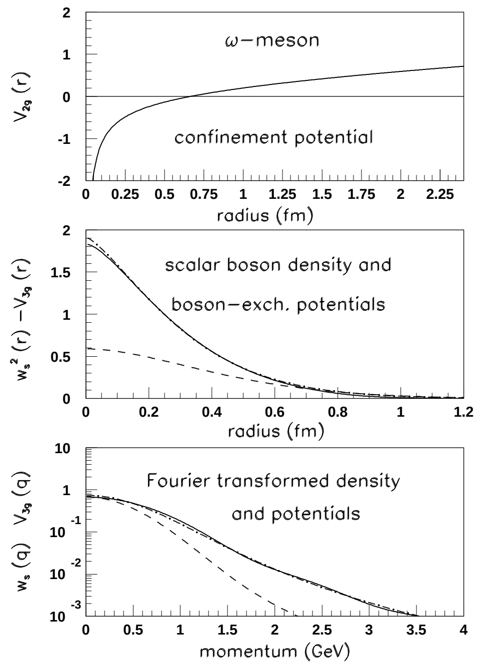

Fig. 2. Radial dependence of a self-consistent solution with $< r^2_{w_s} >^{1/2} = 0.51$ fm (ω meson). Upper part: Confinement potential $V_{2g}(r)$. Second part: Boson density $w^2_s(r)$ (dot-dashed line) and boson-exchange potentials $|V^{s,v}_{3g}(r)|$ given by dashed and solid lines, respectively. Lower part: Fourier transformed boson density $w^2_s(q)$ and potentials $V^{s,v}_{3g}(q)$

In the second part of Fig. 2 the potentials $V_{3g}^{s,v}(r)$ are given together with the boson density $w_s^2(r)$. This shows that the geometrical matching condition (2.17) is fulfilled. The Fourier transformed boson density and potentials are shown in the lower part of fig. 2, indicating very similar features of the system in r- and q-space, as expected. The average width of the momentum distribution of about 1 GeV is significantly smaller than the mass difference between scalar and vector states (see table 1), implying that in the case of hadrons a mixing of both states is small. Results for these solutions are given in Table 1. The obtained root mean square radius is in rather good agreement with the corresponding hadron radius [4].

In Table 2 the corresponding momenta and binding energies are given. The errors in the momenta $< q_{g,f}^2 >^{1/2}$ are mainly due to spacing and cut-off in radius and momentum, which have been estimated by changing the momentum cut-off q_{cut} by ±10 % from a value of about 7 times $< q_g^2 >_s^{1/2}$. The resulting boson and fermion momenta $< q_g^2 >^{1/2}$ and $< q_f^2 >^{1/2}$ are equal within the estimated errors, as required from condition (2.19). Further, both energy-momentum conditions (2.22) and (2.23) are correctly fulfilled. Since there are two states, in addition to the mass-radius constraint (2.21) six conditions had to be fulfilled.

Table 1. Solutions for different fermion-antifermion systems with $\kappa = 1.35$ and $\alpha = 2.14$. Hadron radius R_{had} taken from ref. [4], covalent radius from ref. [5]

binding of massless fermions (mesons) - all quantities in GeV or fm

system	b	M_s	M_v	M_{exp}	$< r_{\psi_s}^2 >^{1/2}$	R_{had}
ω	5.72 10^{-1}	0.78	3.66	0.78	0.65	0.72

binding of massive fermions (atoms) - all quantities in eV or pm

system	b	E_f(1s)	E_f(2s)	$R_{1/2}^{1s}$	$R_{1/2}^{2s}$	R_{Bohr}	R_{cov}
$p - e^-$	115	-13.6	-3.4	34	59	53	31±5
$e^+ - e^-$	230	-6.8	-1.7	68	118	106	

Table 2. Boson and fermion momenta and the corresponding binding energies and masses

binding of massless fermions (mesons) - all quantities in GeV

system	s	$< q_g^2 >^{1/2}$	E_g	$< q_f^2 >^{1/2}$	$< q_f^2 >_{s,v}^{1/2}$	xM_f
ω	0	0.79 ± 0.02	-0.80	0.76±0.04	0.76±0.04	0.78
	1	1.18 ± 0.07	-1.20	1.33±0.14	3.7±0.14	3.66

binding of massive fermions (atoms) - all quantities in keV

system	$< q_g^2 >^{1/2}$	E_g	$< q_f^2 >^{1/2}$	xM_f
$p - e^-$, (2s)	3.9 ± 0.2	-4.0	3.8 ± 0.3	3.9
$p - e^-$, (1s)	6.0 ± 0.5	-6.3	6.6 ± 0.8	6.3
$e^+ - e^-$, (2s)	2.0 ± 0.1	-2.0	1.9 ± 0.2	2.0
$e^+ - e^-$, (1s)	3.0 ± 0.2	-3.2	3.3 ± 0.4	3.2

4 Bound States of Atoms

Bound state solutions composed of hadrons and leptons leading to $p - e^-$ and $e^+ - e^-$ bound states have been studied also. For these cases the mass parameter $\tilde{m}$ is the reduced mass $\tilde{m} = m_1 m_2/(m_1 + m_2)$, where m_i are the masses of the participating particles. A good description of these systems is obtained by assuming the same values of κ and α as needed for hadrons.

The radial properties of the $p - e^-$ system are shown in fig. 3. A similar picture is obtained as for hadronic solutions, with a characteristic linearly rising form of $V_{2g}(r)$ at larger radii (which may be called here stabilizing potential). Further, the boson density and potentials in r- and q-space, $w_s^2(r, q)$ and $V_{3g}^{s,v}(r, q)$, show that relation (2.17) is fulfilled.

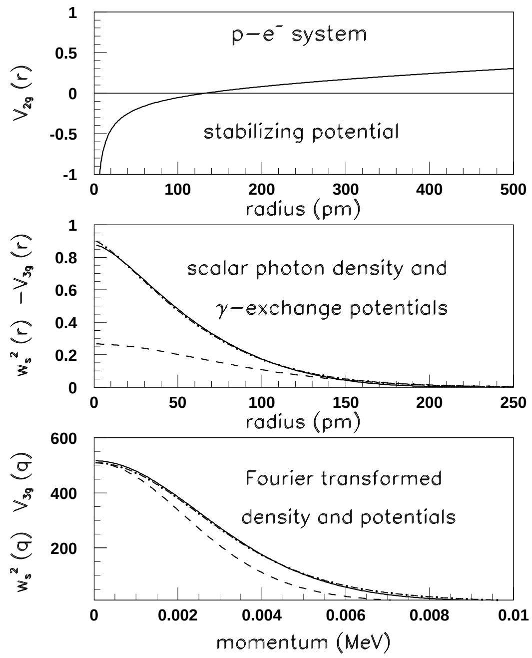

Fig. 3. Radial dependence of a $p - e^-$ bound state solution with $< r^2_{w_s} >^{1/2}$ = 100 pm. Upper part: Stabilizing potential $V_{2g}(r)$. Middle part: Boson density $w_s^2(r)$ (dot-dashed line) and boson-exchange potentials $|V_{3g}^{s,v}(r)|$ given by dashed and solid lines, respectively. Lower part: Fourier transformed boson density $w_s^2(q)$ and potentials $V_{3g}(q)$

However, in the Fourier transformed density and potentials the momentum distribution of a few keV is quite comparable to the difference in average momentum between both states (this is different from hadrons) and indicates an appreciable mixing between scalar and vector states. Using binding energies $E(2s) = (1-x)\ E_s + x\ E_v$ and $E(1s) = (1-x)\ E_v - x\ E_s$ with mixing parameter $x \sim 0.2$, the known energies $E(1s)$ = -13.6 eV and $E(2s)$ = -3.4 eV are well described. The results are given in the lower part of table 1. The resulting root mean square radii $< r^2_{w_{s,v}} >^{1/2}$ for the hydrogen atom are 101 and 54 pm for the 2s and 1s state, respectively, with estimated uncertainties of less than 10 pm. This leads to radii at half maximum $R_{1/2}$ in reasonable agreement with the covariant radius from ref. [5]. For the $e^+ - e^-$ bound system the slope parameter b is a factor of 2 larger, giving rise to root mean square radii of 202 and 108 pm for the 2s and 1s state, respectively.

Inspecting the extracted boson and fermion momenta and energies, the results are given in the lower part of table 2. Different from the case of hadrons, for the vector states also Fourier transformed wave functions and potentials of the form $\psi_v(q) = 4\pi \int r^2 dr\ \psi_v(r) j_o(qr)$ and $V_{3g}^v(q) = 4\pi \int r^2 dr\ V_{3g}^v(r) j_o(qr)$ are needed. This may be due to the fact that the wave functions and interactions $v_v(r)$ are not localized as in hadrons. We see in table 2 that fermion and boson momentum matching is fulfilled in the hydrogen and positronium atom, further that energy-momentum conservation is fulfilled in all cases.

5 Discussion

These results show astonishing features: In a theory, in which only three parameters (κ, b and α) could be adjusted, experimental data of very different systems are well described and further in total about ten constraints had to be fulfilled. This is possible only in a unique theory, in which many quantities are interrelated. In particular, the special form of $V_{2g}(r)$ gives rise to the mass-radius constraint (2.21), which has been checked for hadrons and atoms with binding energies different by more than eight orders of magnitude. Further, the treatment of bosons with only one potential is rather different from that of fermions with two potentials $V_{2g}(r)$ and $V_{3g}(r)$, which have to match geometric properties, which do not depend on the potential $V_{2g}(r)$. Therefore, a matching of boson and fermion properties are possible only, if all definitions are correct. This cannot be taken for granted, since the densities, momenta as well as mass could be defined differently; further, E_o could be assumed different from zero. However, the present definitions appear to be the most simple, natural and logic ones.

So far an explicit bound state description of particles, as described here, has not been found. For atoms bound state energies are well described in Bohr's model of the atom [6, 7] and in the Coulomb potential, an effective potential of fermions only. However, this potential does not allow to extract radial properties of atoms.

Hadrons belong to the domain of the strong interaction, for which quantum chromodynamics (QCD) has been developed, a first order quantum field theory including color, see e.g. [8]. QCD is an effective theory with external parameters adjusted to experimental data, six masses of elementary fermions (quarks), coupling constant and coupling parameters to noncolored fields. Phenomenological potential models have been used to describe hadron bound states, see ref. [3], using quark masses consistent with those extracted from QCD together with an empirical "confinement" potential, which has been attributed to color (see also the discussion of the confinement problem in lattice QCD, ref. [9]). However, it is quite surprising that the confinement potential has a characteristics very similar to $V_{2g}(r)$ (which raises strong doubts that it could be related to color). Because of the close similarity of this potential in both models, in a preliminary version of the present formalism [10] the quark masses could be identified with eigenvalues in the potential $V_{3g}(r)$. This may indicate that in a fundamental theory quarks can be interpreted as effective fermions with masses given by eigenvalues in $V_{3g}(r)$.

6 Conclusion

Starting from quantum field theory a solution of the free bound state problem for microscopic systems has been found, which exhibits severe boundary conditions related to momentum matching, geometry and energy-momentum conservation. In this framework a consistent description of hadronic and atomic bound states is obtained, in which the large differences in mass and radius between these systems are well understood. With only three adjustable parameters and a total of about 10 constraints all definitions and results are strongly interwoven, revealing a well defined mathematical structure and most likely the only closed theory, which could be verified in so much detail. This may not be surprising, since free particle bound states could be the most fundamental objects of nature.

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Competing Interests

The author has declared that no competing interests exist.

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